

Moment Generating Function

The k^{th} moment of X (the k^{th} moment of X about the origin), μ_k , is given by

$$\mu_k = E(X^k) = \sum_{\text{all } x} x^k \cdot f(x)$$

The k^{th} central moment of X (the k^{th} moment of X about the mean), μ'_k , is given by

$$\mu'_k = E((X - \mu)^k) = \sum_{\text{all } x} (x - \mu)^k \cdot f(x)$$

The moment-generating function of X , $M_X(t)$, is given by

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx} \cdot f(x)$$

Theorem 1: $M'_X(0) = E(X)$ $M''_X(0) = E(X^2)$
 $M_X^{(k)}(0) = E(X^k)$

Theorem 2: $M_{X_1}(t) = M_{X_2}(t)$ for some interval containing 0
 $\Rightarrow f_{X_1}(x) = f_{X_2}(x)$

Theorem 3: Let $Y = aX + b$. Then $M_Y(t) = e^{bt} M_X(at)$

Example 1:

Suppose a random variable X has the following probability distribution:

x	$f(x)$
10	0.20
11	0.40
12	0.30
13	0.10

Find the moment-generating function of X , $M_X(t)$.

Example 2:

Suppose the moment-generating function of a random variable X is

$$M_X(t) = 0.10 + 0.15 e^t + 0.20 e^{2t} + 0.25 e^{-3t} + 0.30 e^{5t}.$$

Find the expected value of X , $E(X)$.

Example 3:

Suppose a discrete random variable X has the following probability distribution:

$$f(0) = P(X = 0) = 2 - e^{1/2}, \quad f(k) = P(X = k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \dots$$

a) Find the moment-generating function of X , $M_X(t)$.

b) Find the expected value of X , $E(X)$, and the variance of X , $\text{Var}(X)$.

Example 4:

Let X be a Binomial(n, p) random variable.

Find the moment-generating function of X .

Example 5:

Let X be a geometric random variable with probability of “success” p .

a) Find the moment-generating function of X .

b) Use the moment-generating function of X to find $E(X)$.

Example 6:

a) Find the moment-generating function of a Poisson random variable.

Consider $\ln M_X(t)$. (cumulant generating function)

$$\left(\ln M_X(t) \right)' = \frac{M_X'(t)}{M_X(t)} \quad \left(\ln M_X(t) \right)'' = \frac{M_X''(t) \cdot M_X(t) - [M_X'(t)]^2}{[M_X(t)]^2}$$

Since $M_X(0) = 1$, $M_X'(0) = E(X)$, $M_X''(0) = E(X^2)$,

$$\left(\ln M_X(t) \right)' \Big|_{t=0} = E(X) = \mu_X$$

$$\left(\ln M_X(t) \right)'' \Big|_{t=0} = E(X^2) - [E(X)]^2 = \sigma_X^2$$

b) Find $E(X)$ and $\text{Var}(X)$, where X is a Poisson random variable.