STAT 400
UIUC

Continuous Random Variables
(vs. Discrete RVs)

Stepanov
Dalpiaz
Nguyen

random variables

discrete	**continuous**
probability **mass** function	probability **density** function
p.m.f.	p.d.f.
$p(x) = P(X = x)$	$f(x)$
$\forall x \quad 0 \le p(x) \le 1$	$\forall x \quad f(x) \ge 0$
$\sum_{\text{all } x} p(x) = 1$	$\int_{-\infty}^{\infty} f(x)\, dx = 1$

cumulative distribution function

c.d.f.

$$F(x) = P(X \le x)$$

$$F(x) = \sum_{y \le x} p(y) \qquad\qquad F(x) = \int_{-\infty}^{x} f(y)\, dy$$

expected value

$$E(X) = \mu_X$$

discrete	continuous
$E(X) = \sum_{\text{all } x} x \cdot p(x)$	$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

discrete	continuous
$E(g(X)) = \sum_{\text{all } x} g(x) \cdot p(x)$	$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

variance

$$\mathrm{Var}(X) = \sigma_X^2 = E([X - \mu_X]^2) = E(X^2) - [E(X)]^2$$

discrete

$$\mathrm{Var}(X) = \sum_{\text{all } x} (x - \mu_X)^2 \cdot p(x)$$

$$= \sum_{\text{all } x} x^2 \cdot p(x) - [E(X)]^2$$

continuous

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f(x)\, dx$$

$$= \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx\right] - [E(X)]^2$$

moment-generating function

$$M_X(t) = E(e^{tX})$$

discrete

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x)$$

continuous

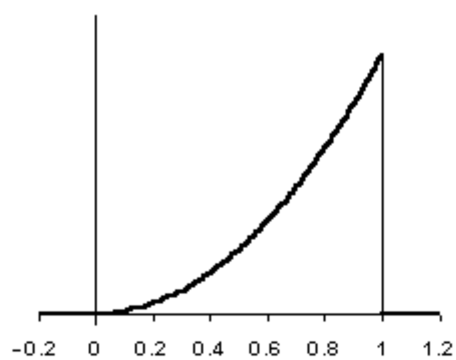
$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

Example 1:

Let X be a continuous random variable with the probability density function

$f(x) = k \cdot x^2$, $\quad 0 < x < 1$,

$f(x) = 0$, $\quad$ otherwise.



a) What must the value of k be so that $f(x)$ is a probability density function?

b) Find the cumulative distribution function $F(x) = P(X \le x)$.

c) Find the probability $P(0.4 \le X \le 0.8)$.

d) Find the median of the distribution of X.

e) Find $\mu_X = E(X)$.

f) Find $\sigma_X = SD(X)$.

g) Find the moment-generating function of X, $M_X(t)$.

h) Find $E(\sqrt{X})$ and $E(\ln X)$.

<u>Example 2</u>:

Suppose a random variable X has the following probability density function:

$$f(x) = \begin{cases} C \cdot e^{-x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) What must the value of C be so that $f(x)$ is a probability density function?

b) Find $\mu_X = E(X)$.

c) Find the cumulative distribution function $F(x) = P(X \leq x)$.

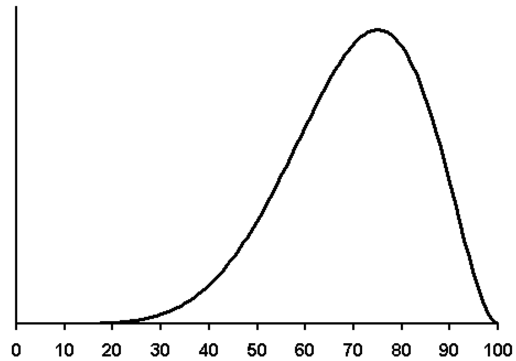
d) Find the median of the probability distribution of X.

e) Find the moment-generating function of X, $M_X(t)$.

f) Find $E(2^X)$.

Example 3:

A simple model for describing mortality in the general population in a particular country is given by the probability density function



$$f(y) = \frac{252}{10^{18}}\ y^6 (100 - y)^2, \qquad 0 < y < 100.$$

a) Verify that $f(y)$ is a valid probability density function.

b) Based on this model, which event is more likely

A: a person dies between the ages of 70 and 80

or

B: a person lives past age 80?

c) Given that a randomly selected individual just celebrated his 60th birthday, find the probability that he will live past age 80.

d) Find the value of y that maximizes $f(y)$ (**mode**).

e) Find the (average) life expectancy.

f) Find the standard deviation of the lifetimes.

Example 4:

Let Y denote a random variable with probability density function given by

$$f(y) = \frac{1}{2} e^{-|y|}, \qquad -\infty < y < \infty. \qquad (\text{double exponential p.d.f.})$$

a) Find the moment-generating function of Y. For which values of t does it exist?

b) Find $E(Y)$.

c) Find $Var(Y)$.

d) Find the cumulative distribution function $F(y) = P(Y \le y)$.

e) Find $E(Y^k)$ for positive integer k.