

Maximum Likelihood Estimation, Method of Moments – Part 1

p.m.f. or p.d.f. $f(x; \theta)$, $\theta \in \Omega$. Ω – parameter space.

Example 1: Suppose $\Omega = \{1, 2, 3\}$ and the p.d.f. $f(x; \theta)$ is

$$\theta = 1: \quad f(1; 1) = 0.6, \quad f(2; 1) = 0.1, \quad f(3; 1) = 0.1, \quad f(4; 1) = 0.2.$$

$$\theta = 2: \quad f(1; 2) = 0.2, \quad f(2; 2) = 0.3, \quad f(3; 2) = 0.3, \quad f(4; 2) = 0.2.$$

$$\theta = 3: \quad f(1; 3) = 0.3, \quad f(2; 3) = 0.4, \quad f(3; 3) = 0.2, \quad f(4; 3) = 0.1.$$

What is the maximum likelihood estimate of θ (based on only one observation of X) if ...

a) $X = 1;$

b) $X = 2;$

c) $X = 3;$

d) $X = 4.$

Likelihood function:

$$L(\theta) = L(\theta; x_1, x_2, \dots, x_n) = \prod_{i=1}^n f(x_i; \theta) = f(x_1; \theta) \cdot \dots \cdot f(x_n; \theta)$$

It is often easier to consider $\ln L(\theta) = \sum_{i=1}^n \ln f(x_i; \theta)$.

Maximum Likelihood Estimator: $\hat{\theta} = \arg \max L(\theta) = \arg \max \ln L(\theta)$.

Method of Moments:

$$E(X) = g(\theta). \quad \text{Set } \bar{X} = g(\tilde{\theta}). \quad \text{Solve for } \tilde{\theta}.$$

Example 2:

Consider a single observation X of a Binomial random variable with n trials and probability of “success” p . That is,

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

a) Obtain the method of moments estimator of p , \tilde{p} .

b) Obtain the maximum likelihood estimator of p , \hat{p} .

Example 3:

Let X_1, X_2, \dots, X_n be a random sample of size n from a Geometric distribution with probability of “success” p , $0 < p < 1$. That is,

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \dots$$

a) Obtain the method of moments estimator of p , \tilde{p} .

b) Obtain the maximum likelihood estimator of p , \hat{p} .