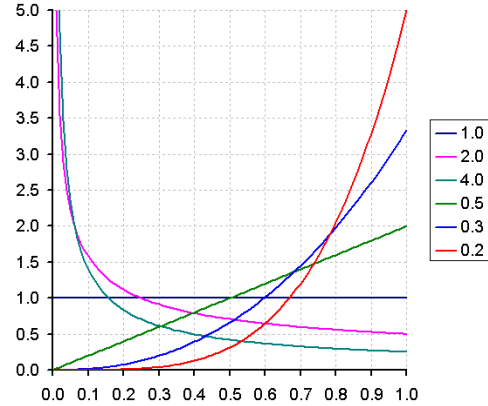


Example 1:

Let X_1, X_2, \dots, X_n be a random sample of size n from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} \cdot x^{1-\theta/\theta} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \theta < \infty.$



Recall: Maximum likelihood estimator of θ is $\hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^n \ln X_i.$

Method of moments estimator of θ is $\tilde{\theta} = \frac{1-\bar{X}}{\bar{X}} = \frac{1}{\bar{X}} - 1. \quad E(X) = \frac{1}{1+\theta}.$

Def An estimator $\hat{\theta}$ is said to be **unbiased for θ** if $E(\hat{\theta}) = \theta$ for all $\theta.$



Accurate but Imprecise
unbiased,

large variance



Inaccurate but Precise
biased,

small variance



Accurate and Precise
unbiased,

small variance

a) Is $\hat{\theta}$ unbiased for θ ? That is, does $E(\hat{\theta})$ equal θ ?

Jensen's Inequality:

If g is convex on an open interval I and X is a random variable whose support is contained in I and has finite expectation, then

$$E [g (X)] \geq g [E (X)].$$

If g is strictly convex then the inequality is strict, unless X is a constant random variable.

b) Is $\tilde{\theta}$ unbiased for θ ? That is, does $E(\tilde{\theta})$ equal θ ?

sample mean

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

sample variance

$$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$$

Example 2:

Let X_1, X_2, \dots, X_n be a random sample of size n from a population with mean μ and variance σ^2 . Show that the sample mean \bar{X} and the sample variance S^2 are unbiased for μ and σ^2 , respectively.

For an estimator $\hat{\theta}$ of θ , define the **Mean Squared Error** of $\hat{\theta}$ by

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2].$$

$$E[(\hat{\theta} - \theta)^2] = (E(\hat{\theta}) - \theta)^2 + \text{Var}(\hat{\theta}) = (\text{bias}(\hat{\theta}))^2 + \text{Var}(\hat{\theta}).$$