

## Confidence Intervals for Means

### Student's t Distribution

Let  $X_1, X_2, \dots, X_n$  be i.i.d.  $N(\mu, \sigma^2)$ .

Let

$$\bar{X} = \frac{\sum X_i}{n} = \frac{X_1 + X_2 + \dots + X_n}{n} \quad (\text{sample mean})$$

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} \quad (\text{sample variance})$$

Then

$\bar{X}$  and  $S^2$  are independent;

$\bar{X}$  has  $N\left(\mu, \frac{\sigma^2}{n}\right)$  distribution;

$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$  has  $N(0, 1)$  distribution;

$\frac{\sum (X_i - \mu)^2}{\sigma^2}$  has  $\chi^2(n)$  distribution;

$\frac{(n-1) \cdot S^2}{\sigma^2} = \frac{\sum (X_i - \bar{X})^2}{\sigma^2}$  has  $\chi^2(n-1)$  distribution;

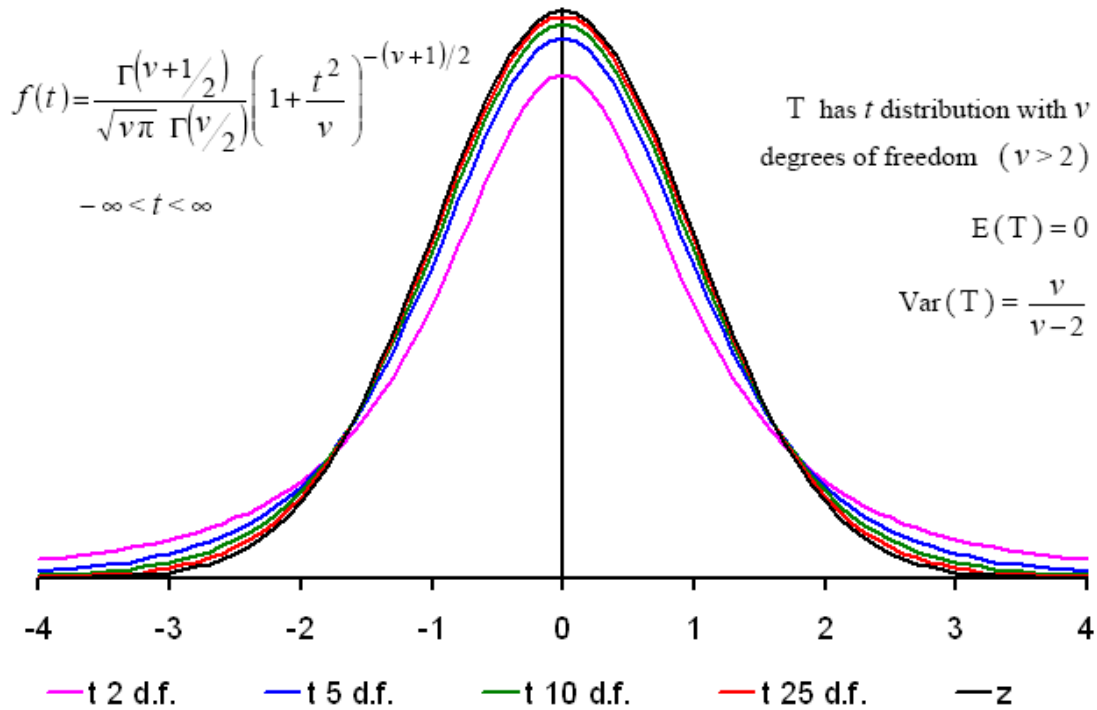
$\frac{\bar{X} - \mu}{S/\sqrt{n}}$  has  $t(n-1)$  distribution.

A  $(1 - \alpha)$  100% confidence interval for the population mean  $\mu$

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \qquad \bar{x} \pm t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

$n - 1$  degrees of freedom

# The $t$ Distribution



**Example 0:**

Let  $Z$  be a  $N(0, 1)$  standard normal random variable.

Then  $X = Z^2$  has a chi-square distribution with 1 degree of freedom.

**Example 1/2:**

Let  $X$  and  $Y$  be two independent  $\chi^2$  random variables with  $m$  and  $n$  degrees of freedom, respectively.

Then  $W = X + Y$  has a chi-square distribution with  $m + n$  degrees of freedom.

**Example 1:**

A manufacturer of TV sets wants to find the average selling price of a particular model. A random sample of 25 different stores gives the mean selling price as \$342 with a sample standard deviation of \$14. Assume the prices are normally distributed. Construct a 95% confidence interval for the mean selling price of the TV model.

**Example 2:**

The following random sample was obtained from  $N(\mu, \sigma^2)$  distribution:

16      12      18      13      21      15      8      17

a) Compute the sample mean and the sample standard deviation.

b) Construct a 95% confidence interval for  $\mu$ .

Interpretation:

- 1) Probability of  $\mu$  being in this interval is 95%.
- 2) Probability of  $\bar{X}$  being in this interval is 95%.
- 3) If we repeat this process many times, 95% of the CIs computed will contain  $\mu$ .

c) Construct a 90% confidence upper bound for  $\mu$ .

d) Construct a 99% confidence lower bound for  $\mu$ .